Ikeda's conjecture on the period of the Ikeda lifting

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§1. Introduction $\quad k \in \mathbb{Z}_{>0}$.

$S_k(\Gamma) = \mathbb{C}$-vector space of cusp forms of wt k w.r.t. Γ.

e.g. $\Gamma = SL_2(\mathbb{Z})$ or $SL_2(\mathcal{O}_K)$ $\quad$ $K \mid \mathbb{Q}$ totally real ...

$Sp_n(\mathbb{Z})$

$\Gamma = SL_2(\mathbb{Z})$

$f \in S_k(SL_2(\mathbb{Z}))$ a normalized Hecke eigenform $a_1(f) = 1$.

$$= \sum_{N \geq 1} a_N(f) \cdot e^{2\pi\sqrt{-1}\cdot N \cdot z} \qquad z \in \mathfrak{h} \subset \mathbb{C}.$$

χ a Dirichlet character,
$$L(s,f,\chi) = \prod_{p \text{ prime}} \left\{ (1 - \chi(p)\cdot\alpha_p \cdot p^{\frac{(k-1)}{2} - s})(1 - \chi(p)\alpha_p^{-1}\cdot p^{\frac{(k-1)}{2} - s})\right\}^{-1}$$

where $\alpha_p \in \mathbb{C}$ s.t. $\quad \alpha_p + \alpha_p^{-1} = a_p(f)\cdot p^{-(\frac{k-1}{2})}$

$$L(s,f,\mathrm{Ad},\chi) = \prod_p \left\{ (1-\chi(p)p^{-s})(1-\chi(p)\alpha_p^2 p^{-s})(1-\chi(p)\alpha_p^{-2}p^{-s}) \right\}^{-1}$$

For $\underset{1}{\chi}$ principal $\Rightarrow$ get $L(s,f)$, $L(s,f,\mathrm{Ad})$

$$\langle f, f\rangle = \int_{SL_2(\mathbb{Z})\backslash\mathfrak{h}} |f(z)|^2 y^k \frac{dx\,dy}{y^2} \qquad z = x + \sqrt{-1}y.$$

Rmk (Sturm) $\quad 1 \leq m \leq k-1$, $\quad \chi(-1) = (-1)^{m-1}$.

$$\Rightarrow \frac{L(m,f,\mathrm{Ad},\chi)}{\pi^{k+2m-1}\langle f,f\rangle} \in \mathbb{Q}(f,\chi) \quad \text{a totally real fld over } \mathbb{Q}.$$

$\left(\begin{matrix} s \leftrightarrow 1-s \\ \text{centered} \\ \text{at } \frac{1}{2} \end{matrix}\right)$?

(a consequence of theta lifting?)

$F/\mathbb{Q}$ a real quadratic

$\cup$

$\mathcal{O}_F$

- $\Gamma = SL_2(\mathcal{O}_F)$.

For simplicity, we assume $h(F)^+ = 1$

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$\hat{f} \in S_k(SL_2(\mathcal{O}_F))$ the Doi - Naganuma lift of f

$$L(s,\hat{f}) = L(s,f)\,L(s,f,\underbrace{\chi_F}_{\text{Kronecker character of } F.})$$

<u>Fact</u> 1: (Doi - Hida - Ishii)

$$\frac{2^{2k}\langle \hat{f}, \hat{f}\rangle}{D_F^k \langle f, f\rangle^2} = \frac{L(1, f, \mathrm{Ad}, \chi_F)}{\pi^{k+1}\langle f, f\rangle} \in \mathbb{Q}(f,\chi)$$

Doi-Hida-Ishii Conj (Congruence between Hilbert modular forms ...)

K sufficiently large alg. number field

$\mathfrak{P}$ a prime ideal in K.

$\Rightarrow$ the following two are equivalent

(i) $\mathfrak{P} \mid$ (the numerator <u>RHS</u> above)

(ii) $\exists\, g \in S_k(SL_2(\mathcal{O}_F))$ not coming from D-N lift,

s.t. $g \equiv \hat{f} \mod \mathfrak{P}$

<u>Rmk</u> (i) $\Rightarrow$ (ii) holds under some assumptions (Urban, Katsurada)

The aim of this talk is to study analogous situations for Siegel modular forms:

- Petersson inner product of Siegel modular forms (the <u>Ikeda lift</u>.)
- Congruence of ____________.

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§2. Ikeda lifting and the Ikeda conj.

$n \in \mathbb{Z}_{>0}$, $Sp_n(\mathbb{Z}) := \{ g \in GL_{2n}(\mathbb{Z}) \mid {}^t g \underbrace{\begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix}}_{= J} g = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} \}$

$F \in S_k(Sp_n(\mathbb{Z}))$ a Hecke eigenform

$$L(s, F, St) := \prod_p \left\{ (1-p^{-s}) \prod_{i=1}^{n} (1-\beta_p^{(i)} p^{-s})(1-(\beta_p^{(i)})^{-1} p^{-s}) \right\}^{-1}$$

$\{\beta_p^{(i)}\}_{1 \le i \le n}$: Satake parameters of F.

(at least over unramified places)

$$\boxed{\langle F, F \rangle = \int_{Sp_n(\mathbb{Z}) \backslash \mathfrak{H}_n} |F(z)|^2 \det(Y)^k \frac{dx\,dy}{\det(Y)^{n+1}}}$$

$Z = X + iY$,

$\mathfrak{H}_n = \{ Z \in Sym_n(\mathbb{C}) \mid \underbrace{Im(Z)}_{= Y} > 0 \}$. $Z = X + \sqrt{-1}Y$

Rmk: (Böcherer, Mizumoto)

$Sym_n^*(\mathbb{Z})_+ := \{ T \in Sym_n(\mathbb{Q}) \mid 2T \text{ pos. def. even integral} \}$

$\forall\ T \in Sym_n^*(\mathbb{Z})_+ \qquad 1 < m \le k-n \qquad m \equiv 0 \mod 2$

$$\Rightarrow \boxed{|c_F(T)|^2 \cdot \frac{L(m, F, St)}{\pi^{nk + (n+1)m - \frac{(n+1)n}{2}} \langle F, F \rangle} \in \mathbb{Q}(F).}$$

Fact 2 (Ikeda) Fix $n, k \in 2\mathbb{Z}_{>0}$ s.t. $k > n+1$

$f \in S_{2k-n}(SL_2(\mathbb{Z}))$: a normalized Hecke eigenform

$\Rightarrow \exists\ I_n(f) \in S_k(Sp_n(\mathbb{Z}))$ a Hecke eigenform s.t.

$$L(s, I_n(f), St) = \zeta(s) \prod_{i=1}^{n} L(s+k-i, f).$$

Rmk. (i) $n=2 \Rightarrow I_2(f) =$ Saito-Kurokawa lifting of f.

$S^+_{k-\frac{1}{2}}(\Gamma_0(4)) \xrightarrow[\simeq]{\text{Shimura}} S_{2k-2}(SL_2(\mathbb{Z}))$

$\cap$ $SL_2(\mathbb{Z})$ $\leftarrow$ Eichler-Zagier , Maass $S_k(Sp_2(\mathbb{Z}))$.

Kohnen's plus space $\qquad \searrow \quad J^{cusp}_{k,1}(SL_2(\mathbb{Z}) \ltimes (\mathbb{Z}^2 \times \mathbb{Z})) \longrightarrow$

the composition gives $\quad S_{2k-2}(SL_2(\mathbb{Z})) \longrightarrow S_k(Sp_2(\mathbb{Z}))$.

(ii) $I_n(f)$ can be constructed by $\tilde{f} \in S^+_{k-\frac{(n-1)}{2}}(\Gamma_0(4))$

a Hecke eigenform corresponding to f via the Shimura corresp.

(A CAP lifting)

$\{\beta^{(i)}_p\}$: Satake parameters of $I_n(f)$

$$\Rightarrow \quad \beta^{(i)}_p = \begin{cases} \alpha_p\, p^{i-\frac{1}{2}} & \text{if } 1 \le i \le \frac{n}{2} \\ \alpha_p^{-1}\, p^{i-\frac{n+1}{2}} & \text{o/w} \end{cases} \quad \Rightarrow \quad |\beta^{(i)}_p| \ne 1.$$

The image of the Ikeda lifting is a CAP form (under Ramanujan's conj.)

Question: $\langle I_n(f), I_n(f) \rangle = ?$ $\quad \overset{\text{leads to}}{\rightsquigarrow}$ Ikeda's Conj.

$$\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$$

$$\underline{\xi}(s) = \Gamma_{\mathbb{C}}(s)\,\zeta(s)$$

$$\Lambda(s,f) = \Gamma_{\mathbb{C}}(s) \cdot L(s,f).$$

$$\tilde{\Lambda}(s,f,\mathrm{Ad}) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+2k-n-1)\,L(s,f,\mathrm{Ad})$$

Rmk: $\tilde{\xi}(2i) = \dfrac{|B_{2i}|}{2i} \in \mathbb{Q}^\times$

$$1 \le i \le k-\frac{n}{2}-1 \quad \Rightarrow \quad \frac{\tilde{\Lambda}(2i-1,f,\mathrm{Ad})}{\langle f, f \rangle} \in \mathbb{Q}(f).$$

Ikeda Conj.: $\quad f,\ \tilde{f} \quad I_n(f)$ as above,

wt $\quad 2k-n, \quad k-\frac{(n-1)}{2}, \quad k$

1 , 1 , n

$$\Rightarrow \frac{\langle f, f\rangle \langle I_n(f), I_n(f)\rangle}{\langle \tilde{f}, \tilde{f}\rangle} = 2^{-\alpha(n,k)} \Lambda(k,f) \prod_{i=1}^{n/2} \tilde{\Lambda}(2i-1, f, \mathrm{Ad})\, \tilde{\xi}(2i)$$

where $\alpha(n,k) \in \mathbb{Z}_{>0}$,
$\quad\;\; (n-1)(k-\frac{n}{2}+1)$.

Main Thm (Katsurada - K.)

Ikeda's Conj is true for all $n \in 2\mathbb{Z}_{>0}$.

Corollary: $\forall$ $D<0$ fundamental discriminant

$$\frac{\langle I_n(f), I_n(f)\rangle}{\langle f, f\rangle^{n/2}} = 2^{-\beta(n,k)} \frac{c_{\tilde{f}}(|D|)^2 \Lambda(k,f)\, \xi(n)}{|D|^{k-\frac{(n+1)}{2}} \tilde{\Lambda}(k-\frac{n}{2}, f, \chi_D)} \cdot \times \prod_{j=1}^{\frac{n}{2}-1} \frac{\tilde{\xi}(2j)\, \tilde{\Lambda}(2j+1, f, \mathrm{Ad})}{\langle f, f\rangle}$$

This is a consequence of Kohnen-Zagier's relation

$$\frac{c_f(|D|)^2}{\langle \tilde{f}, \tilde{f}\rangle} = \frac{2^{k-n/2-1} |D|^{k-\frac{(n+1)}{2}} \Lambda(k-\frac{n}{2}, f, \chi_D)}{\langle f, f\rangle}$$

(Obtain congruences between CAP and non-CAP forms.)